# OPTIMAL EXECUTION IN A GENERAL ONE-SIDED LIMIT-ORDER BOOK

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# Part 1

basic models, problem specification

# Outline

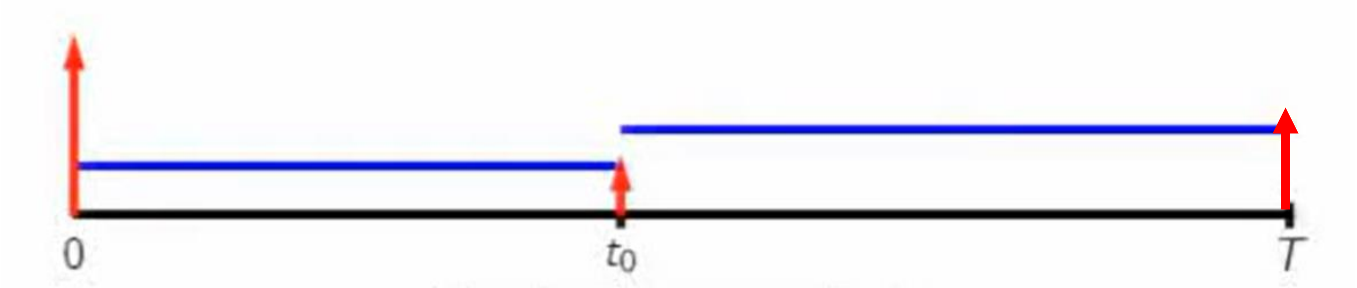
- introduction
- basic model
- problem specification
    - Theorem 3.1
    - Theorem 3.2

# Introduction

- consider optimal execution over a fixed time interval of a large asset **purchase** in the face of a **one-sided limit-order book**
- assume the ask price (best ask price) is **a continuous martingale with two adjustments :**
  - orders consume a part of the limit-order book, and this increase the ask price for subsequent orders
  - **resilience** in the limit-order book causes the effect of these prior orders to decay over time
- there is **no permanent effect** from the purchase we model, but the temporary effect requires infinite time to completely disappear.

# Introduction

- show that the optimal execution strategy **consists of three lump purchases**, and between these lump purchase, the optimal strategy purchases at a **constant rate matched to the limit-order book recovery rate (its resilience),** so that the ask price minus its martingale component remains constant (section 4)




- Goal : Minimize total cost of purchase

The present paper is inspired by Obizhaeva and Wang (2005)

# Review

- Bertsimas & Lo (1998) : Trade on discrete time with permanent / temporary linear price impact, and calculate strategy by using dynamic programming.

$$Min\ E[\textstyle\sum_{t=1}^{T} P_t S_t], \quad P_t = P_{t-1} + \theta S_t + \varepsilon_t, \quad V_t(P_{t-1}, W_t) = Min\ E_t[P_t S_t + V_{t+1}(P_t, W_{t+1})]$$

- Almgren & Chriss (2000) : Trade on discrete time with permanent / temporary linear price impact. Take variance of cost function into account (risk aversion).

$$\min_x \big(E(x) + \lambda V(x)\big)$$

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g\left(\frac{n_k}{\tau}\right) \rightarrow \tilde{S}_k = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

$$\frac{\partial \mathrm{U}}{\partial x_j} \rightarrow x_j = \frac{\sinh\big(\kappa(T - t_j)\big)}{\sinh\big(\kappa T\big)} X$$

# Review

- Obizhaeva & Wang (2005) : price impact of trade will change security's supply and demand (limit-order book, resilience), and the optimal strategy involves both discrete and continuous trades.

Proposition 3 $$J_t = (F_t + s/2)X_t + \lambda X_0 X_t + \alpha_t X_t^2 + \beta_t D_t + \gamma_t D_t^2$$

$$D_t = A_t - V_t - s/2$$

$$x_0 = x_T = \frac{X_0}{\rho T + 2}, \quad \mu_t = \frac{\rho X_0}{\rho T + 2} \quad \forall\ t \in (0,\ T)$$

- Alfonsi, Fruth and Schied (2010) : based on Obizhaeva & Wang, with more general shape of limit order book.

# Basic Model

Section 2

# Basic model

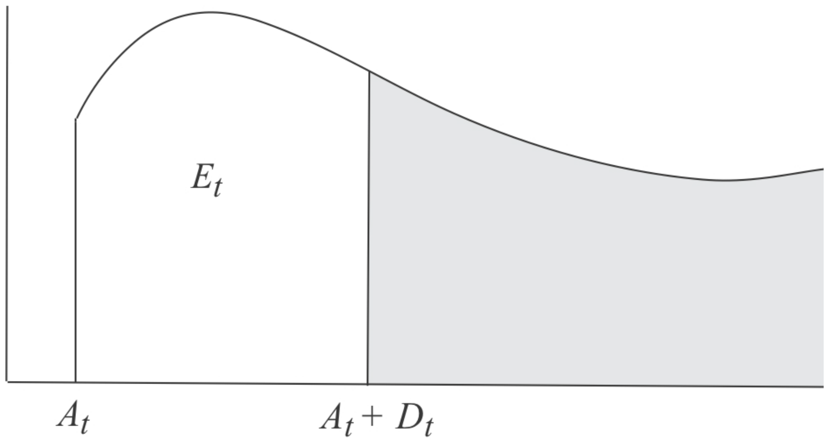
- T : total trading time , $0 \leq t \leq T$
- $\bar{X}$ : total trading volume
- $X_t$ : cumulative purchase up to time t
  - $X_{0-} = 0$, $X_T = \bar{X}$ (nondecreasing, **right continuous**)
- $A_t$: best ask price in the absence of our trade, which is also continuous nonnegative martingale.

# Basic model

- $\mu$ : The shadow order book to the right of A(t), which represent the distribution of sell order
  - M is some extended positive real number
  - if B is a measurable subset of [0,M), then at time $t \geq 0$ the number of limit orders with prices in B+$A_t$ $\triangleq \{b + A_t; b \in B\}$ is μ(B)
- F(x) : shadow limit-order book, which is **left-continuous** cumulative distribution function
  - F(x) ≜ $\mu\big([0, x)\big), x \geq 0$ :
- h(x) : resilience function. Defined on [0, ∞) with h(0)=0, and is strictly increasing and locally Lipschitz
- h(0) = 0 , h(∞) ≜ $\lim_{x \to \infty} h(x) > \frac{\bar{X}}{T}$

# Basic model

- $E_t$ ： residual effect process, which is a unique nonnegative **right-continuous** finite-variation adapted process E satisfying :
  - $E_t = X_t - \int_0^t h(E_s)ds,\ 0 \le t \le T$
  - $E_{0-} = 0,\ \Delta X_t = \Delta E_t$
- Ψ(y) : left continuous inverse of F
  - Ψ(y) ≜ $\sup\{x \ge 0 \mid F(x) < y\}$ , y>0
  - Ψ(0) ≜ Ψ(0+)= 0 (∵ F(x) > 0 for every x>0)




- Ask price in the presence of large investor is defined to be $A_t + D_t$, where :
  - $D_t \triangleq \Psi(E_t)$ , $0 \le t \le T$

# Cost function

suppose $A_t \equiv 0$, and no purchase have been made before :

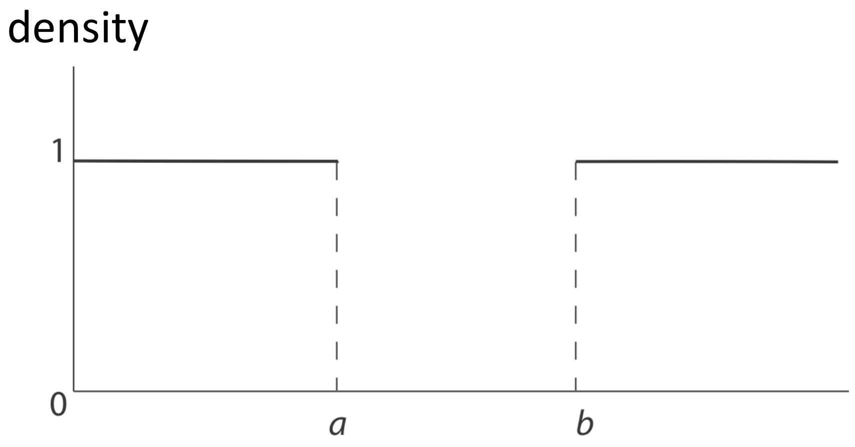
- The cost of purchasing all shares at prices in [0,x) :
  - $\rho(x) \triangleq \int_{[0,x)} \xi dF(\xi)\,, \quad x \geq 0$
- The cost of purchasing y shares is :
  - $\phi(y) \triangleq \rho(\Psi(y)) + [y - F(\Psi(y))]\Psi(y), \quad y \geq 0$ and $\phi(0) = 0$
  - first term : purchasing all shares in the interval [0,x)
  - second term : lump purchase at price $\Psi(y)$

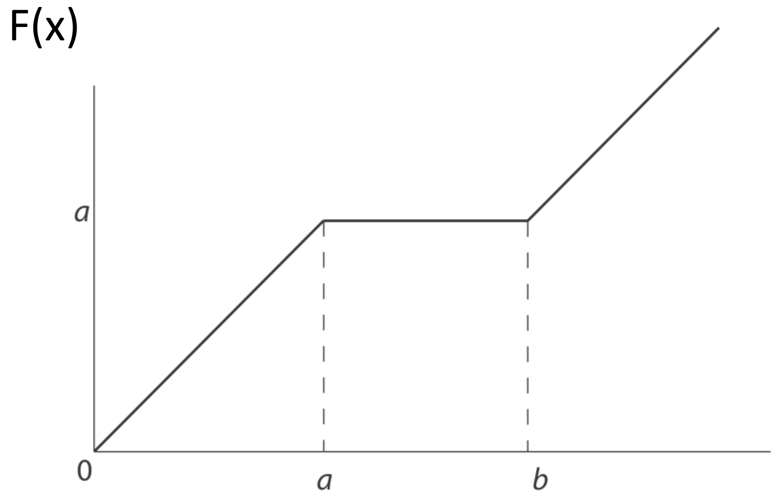
# Example 2 (Modified block order book)

- $F(x) = \begin{cases} x, & 0 \le x \le a \\ a, & a \le x \le b \\ x - (b - a), & b \le x < \infty \end{cases}$

- $\Psi(y) = \begin{cases} y\,, & 0 \le y \le a \\ y + b - a, & a < y < \infty \end{cases}$

- F(Ψ(y)) = y for all $y \ge 0$



$$\rho(x) \triangleq \int_{[0,x)} \xi dF(\xi), \qquad x \geq 0$$

$$\phi(y) \triangleq \rho(\Psi(y)) + [y - F(\Psi(y))]\Psi(y)$$

$$\Psi(y) = \begin{cases} y, & 0 \leq y \leq a \\ y + b - a, & a < y < \infty \end{cases}$$

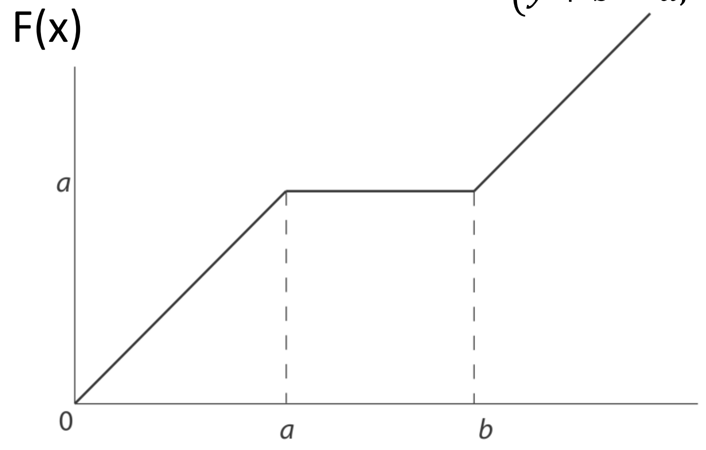
# Example 2 (Modified block order book)

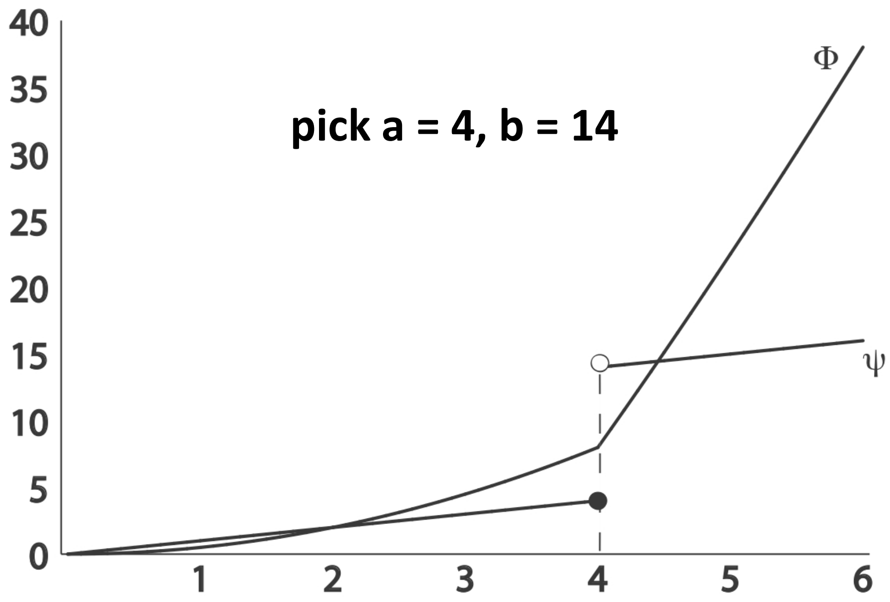
- $$\rho(x) = \begin{cases} \frac{1}{2}x^2, & 0 \leq x \leq a \\ \frac{1}{2}a^2 & a \leq x \leq b \\ \frac{1}{2}(x^2 + a^2 - b^2), & b \leq x < \infty \end{cases}$$

- $$\phi(y) = \begin{cases} \frac{1}{2}y^2, & 0 \leq y \leq a \\ \frac{1}{2}((y + b - a)^2 + a^2 - b^2), & a \leq y < \infty \end{cases}$$

- $\phi(y)$ is convex with subdifferential :

- $$\partial\phi(y) = \begin{cases} \{y\}, & 0 \leq y < a \\ [a, b], & y = a \\ \{y + b - a\}, & a < y < \infty \end{cases}$$



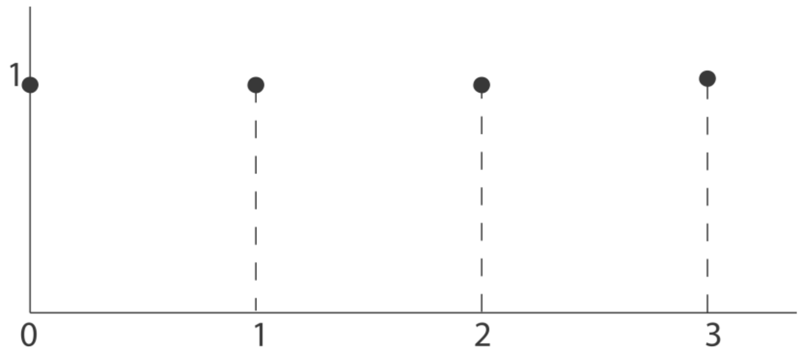



# Example 3 (Discrete order book)

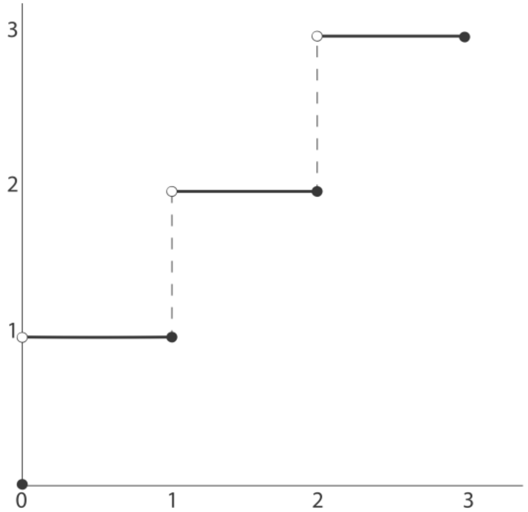
- F(x) = $\sum_{i=0}^{\infty} I_{(i,\infty)}(\mathrm{x}), \quad \mathrm{x} \geq 0$
- Ψ(y)= $\sum_{i=1}^{\infty} I_{(i,\infty)}(\mathrm{y}), \quad \mathrm{y} \geq 0$
- where F(j) = j, F(j+) = j+1, Ψ(j+1) = j, Ψ(j+) = j
- F(Ψ(j)+ ) = j , Ψ(F(j)+) = j

- for $\mathrm{k} \geq 1$ and $\mathrm{k} < \mathrm{y} \leq \mathrm{k} + 1$, Ψ(y)= k

density




F(x)

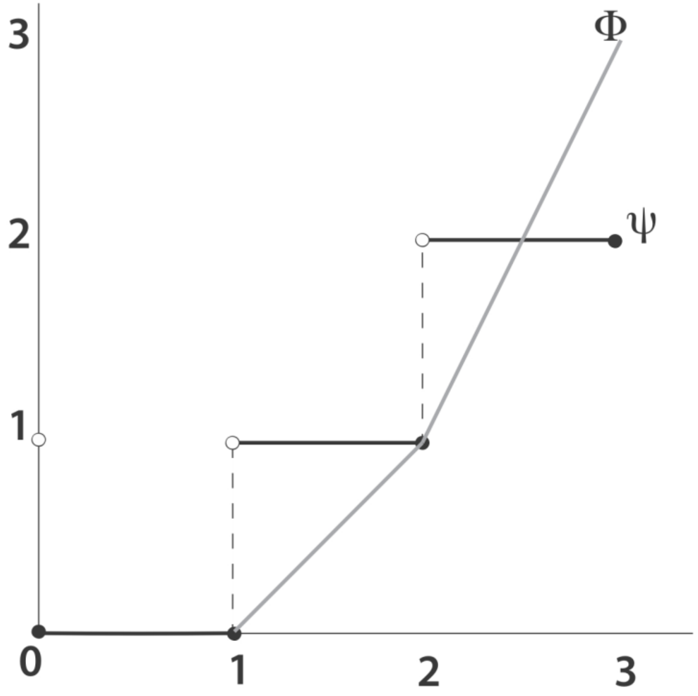


$\rho(x) \triangleq \int_{[0,x)} \xi dF(\xi), \quad x \geq 0$

$\phi(y) \triangleq \rho(\Psi(y)) + [y - F(\Psi(y))]\Psi(y)$

# Example 3 (Discrete order book)

- $\rho(x) = \sum_{i=0}^{\infty} i I_{(i,\infty)}(x)$
  - in particular, $\rho(0) = 0$
  - and for integers $k \geq 1$ and $k - 1 < x \leq k$, $\rho(x) = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}$
- $\phi(y) = \rho(\Psi(y)) + [y - F(\Psi(y))]\Psi(y)$
  - $\rho(\Psi(y)) = \frac{k(k-1)}{2}$ (for $k < y \leq k+1$, $\Psi(y) = k$)
  - lump purchase : $[y - F(\Psi(y))]\Psi(y) = k(y-k)$
  - we get : $\phi(y) = \sum_{k=1}^{\infty} k\left(y - \frac{1}{2}k - \frac{1}{2}\right) I_{(k,k+1]}(y)$
- $\phi$ is convex, with differential :
  - $\partial\phi(y) = [\Psi(y), \Psi(y+)]$, for all $y \geq 0$
  - $\phi'(y) = \Psi(y) = k$, for all $y \geq 0$



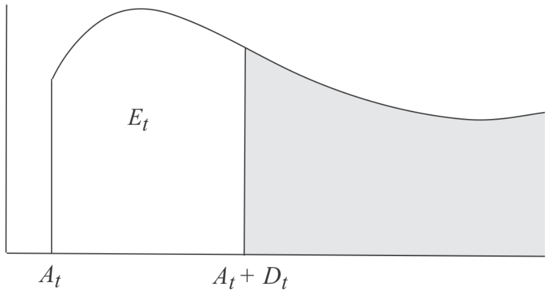
$$\phi(y) \triangleq \rho(\Psi(y)) + [y - F(\Psi(y))]\Psi(y)$$

# Define Cost & strategy

$$E_t = X_t - \int_0^t h(E_s)ds,$$

- decompose strategy X into its continuous and pure jump parts :
  - $X_t = X_t^c + \sum_{0 \le s \le t} \Delta X_s$
- investor pays price $A_t D_t$ for infinitesimal purchase at time t
  - total cost of these purchase : $\int_0^T (A_t + D_t)\, dX_t^c$
- for lump purchase :
  - $\Delta X_t = \Delta E_t$
  - cost of purchase $\Delta X_t$ : $A_t \Delta X_t + \phi(E_t)$- $\phi(E_{t-})$
- total cost function :
  - C(X) = $\int_0^T (A_t + D_t)\, dX_t^c + \sum_{0 \le t \le T} [A_t \Delta X_t + \phi(E_t) - \phi(E_{t-})]$

$$= \int_0^T (D_t) dX_t^c + \sum_{0 \le t \le T} [\phi(E_t) - \phi(E_{t-})] + \int_{[0,T]} A_t\, dX_t$$




**Our Goal is to determine strategy X that minimizes E [C(X)]**

# Problem simplifications

Section 3 (Rewrite cost function)

# Rewrite cost function

- C(X) = $\int_0^T (A_t + D_t\ )\, dX_t^c + \sum_{0\le t\le T}[A_t \Delta X_t +\ \phi(E_t) - \phi(E_{t-})]$

  $= \int_0^T (D_t) dX_t^c + \sum_{0\le t\le T}[\ \phi(E_t) - \phi(E_{t-})]\ + \int_{[0,T]} A_t\, dX_t$
- By integration by parts :
  - $\int_{[0,T]} A_t\, dX_t = \left(\mathrm{A_t X_t}|_0^{\mathrm{T}}\right) - \int_0^T (X_t) dA_t = A_T X_T - A_0 X_0 - \int_0^T (X_t) dA_t$
  - and $\because$ **E ($\int_0^T (X_t) dA_t$) $= 0$ (martingale),** E $(A_T X_T) = \bar{X} A_0$ , E$(A_0 X_0)$ = 0

    $\therefore$ EC(X) = $\int_0^T (D_t) dX_t^c + \sum_{0\le t\le T}[\ \phi(E_t) - \phi(E_{t-})] + \bar{X} A_0$
- minimization of C(X) is equal to min($\int_0^T (D_t) dX_t^c + \sum_{0\le t\le T}[\ \phi(E_t) - \phi(E_{t-})]$ )
- 2 theorems for minimization problem simplification

$$C(X) = \int_0^T (A_t + D_t)\, dX_t^c + \sum_{0\le t\le T}[A_t \Delta X_t + \phi(E_t) - \phi(E_{t-})]$$

$$= \int_0^T (D_t) dX_t^c + \sum_{0\le t\le T}[\phi(E_t) - \phi(E_{t-})] + \int_{[0,T]} A_t\, dX_t$$

# Theorem 3.1

- do not allow the agent to make intermediate sells in order to achieve the ultimate $\bar{X}$ shares, because doing so would not decrease the cost
- proof :

suppose the agent has strategy Y, which is non-decreasing right-continuous adapted process with $Y_{0-} = 0$, $X_T - Y_T = \bar{X}$ **(didn't modeled the limit buy order book)** :

$$C(X,Y) \ge \int_0^T (D_t)\, dX_t^c + \sum_{0\le t\le T}[\phi(E_t) - \phi(E_{t-})] + \int_{[0,T]} A_t\, dX_t - \int_{[0,T]} A_t\, dY_t$$

# Theorem 3.1

- with both buy / sell strategy :

$$C(X,Y) \geq \int_0^T (D_t)\, dX_t^c + \sum_{0\leq t\leq T}[\phi(E_t) - \phi(E_{t-})] + \int_{[0,T]} A_t\, dX_t - \int_{[0,T]} A_t\, dY_t$$

- By integration by parts :
  - $\int_{[0,T]} A_t\, dX_t - \int_{[0,T]} A_t\, dY_t = \mathrm{A_T}(\mathrm{X_T} - \mathrm{Y_T}) - \mathrm{A_0}(\mathrm{X_{0-}} - \mathrm{Y_{0-}}) - \int_0^T (X_t - Y_t) dA_t$
  - and **E($\int_0^T (X_t - Y_t) dA_t) = 0$ (martingale),** so

  $EC(X,Y) \geq E\int_0^T (D_t)\, dX_t^c + \mathrm{E}\sum_{0\leq t\leq T}[\phi(E_t) - \phi(E_{t-})] + \bar{X}A_0 \geq EC(X)$

# Theorem 3.2

- Assume without loss of generality that $\mathrm{A_t} \equiv 0$, the cost of using strategy $X_t, 0 \leq t \leq T$:

  C(X) $= \int_0^T (D_t) dX_t^c + \sum_{0\leq t\leq T}[\, \phi(E_t) - \phi(E_{t-})] \;= \phi(E_T) + \int_0^T D_t h(E_t) dt$

- proof.
  - step 1 : $\partial\phi(y) = [\Psi(y), \Psi(y+)]$
  - step 2 : $\phi(E_T) \;= \int_0^T D_t dX_t^c \;- \int_0^T D_t h(E_t) dt \;+ \sum_{0\leq t\leq T}[\, \phi(E_t) - \phi(E_{t-})]$

# Part 2

Strategy solution, Conclusion

# Outline

- Strategy solution
  - Type A strategy
  - Type B strategy
- Conclusion

# Strategy

Type A & Type B

$$E_t = X_t - \int_0^t h(E_s)ds, \qquad 0 \le t \le T$$

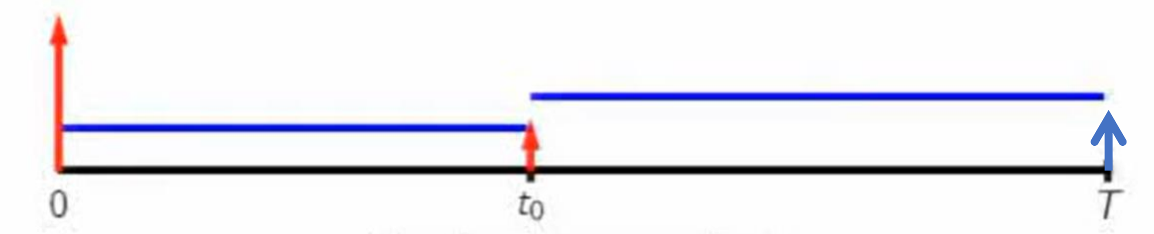
Ψ(y) : left continuous inverse of F

# Optimization Problem

- Goal : minimize the expected cost $\phi(E_T) + \int_0^T D_t h(E_t)dt$
- Two solution :
  - Type B (the optimal one):

    $X_0 = E_0$, then buys $dX_t = h(E_0)dt$ up to time $t_0$, then buys another lump at time $t_0$, subsequently trades again at a constant rate $dX_t = h(E_{t0})dt$ until time T, and finally buy the remaining shares at T.
  - Type A (special case of Type B):

    if $g(y) \triangleq y\Psi(h^{-1}(y))$ is convex, then there exists a Type A purchasing strategy that minimizes C(X) over all purchasing strategies X, where the purchase at time $t_0$ consists of 0 shares.



# Type A Strategy

- Goal : minimize the expected cost $\phi(E_T) + \int_0^T D_t h(E_t)dt$
  - Type A (special case of Type B):
    - if $g(y) \triangleq y\Psi(h^{-1}(y))$ is convex, then there exists a Type A purchasing strategy that minimizes C(X) over all purchasing strategies X, where the purchase at time $t_0$ consists of 0 shares.
  - the cost strategy can be rewrite :
    - C(X) = $\phi(E_T) + \int_0^T D_t h(E_t)dt = \phi(E_T) + \int_0^T g(h(E_t))dt$, where $g(y) \triangleq y\Psi(h^{-1}(y))$
    - $C(X^A) = \phi(E_T^A) + Tg\left(h\left(X_0^A\right)\right) = \phi(E_T^A) + Tg\left(h\left(h^{-1}(\frac{\bar{X}-E_T^A}{T})\right)\right) = \phi(E_T^A) + Tg\left(\frac{\bar{X}-E_T^A}{T}\right)$
  - only when g is convex, we can **use Jensen's inequality** to prove :
    - $\phi(E_T) + \int_0^T g(h(E_t))dt \geq \phi(E_T\ ) + Tg\left(\frac{\bar{X}-E_T}{T}\right)$
  - define G(e) = $\phi(e) + Tg\left(\frac{\bar{X}-e}{T}\right)$, we can find the e*($E_T^A$) that minimize G
    - $X_0^A = h^{-1}(\frac{\bar{X}-E_T^A}{T})$
    - purchase continuously with rate $h\left(X_0^A\right)$
    - $X_T^A = \bar{X} - X_0^A - h\left(X_0^A\right)T$

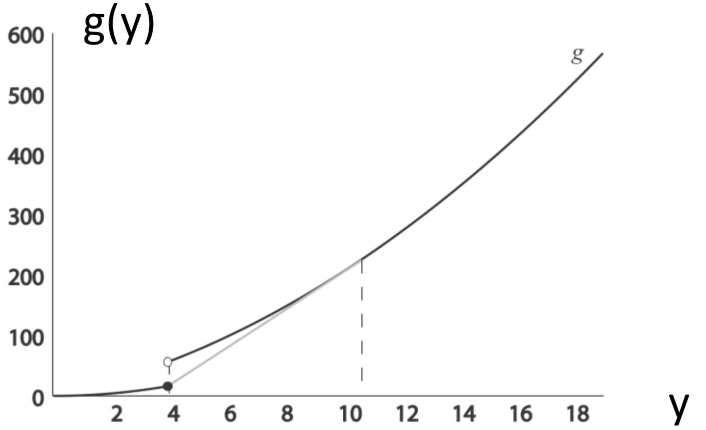
# Type B strategy

- In the absence of the assumption that g is convex, there exists a Type B purchasing strategy that minimizes C(X) over all purchasing strategies X.
- define **convex hull of g**, defined by :
  - $\hat{g}(y) \triangleq \sup\{l\ (y){:}\ l\ is\ an\ affine\ function\ and\ l(\eta) \leq g(\eta) \forall \eta \in [0, \bar{Y}]\}$
  - $\hat{g}(0)$ = g(0) = 0, $\hat{g}(\bar{Y})$ = g($\bar{Y}$)
  - if $y^* \in (0, \bar{\mathrm{Y}})$ which satisfies $\hat{g}(y^*) < g(y^*)$, then exists unique l below g
  - $0 \leq \alpha < y^* < \beta \leq \bar{Y}$ :

    l($\alpha$) = $\hat{g}(\alpha)$ = g($\alpha$), l($\beta$) = $\hat{g}(\beta)$ = g($\beta$)

    l(y) = $\hat{g}(y)$ < g(y), $\alpha < y < \beta$

  (prove in Appendix C)



$$C(X) = \phi(E_T) + \int_0^T g(h(E_t))dt$$

# Type B strategy

- $\hat{C}(X) \triangleq \phi(E_T) + \int_0^T \hat{g}\big(h(E_t)\big)dt$
- we obviously have $\hat{C}(X) \leq C(X)$
- By Jensen's Inequality (12_20 pg, 14) :
  - $\hat{C}(X) \geq \phi(E_T) + T\hat{g}\left(\frac{\bar{X}-E_T}{T}\right)$
- This lead us to consider minimization of the function $\widehat{G}$ :
  - $\widehat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X}-e}{T}\right)$
- prove that :
  - $C(X^B) = \widehat{G}(e^*)$

$$C(X) = \phi(E_T) + \int_0^T g(h(E_t))dt$$

$$\hat{C}(X) \triangleq \phi(E_T) + \int_0^T \hat{g}\big(h(E_t)\big)dt$$

$$g(y) \triangleq y\Psi(h^{-1}(y))$$

# Type B strategy

- $C(X^B) = \hat{G}(e^*) = \phi(e^*) + T\hat{g}\left(\frac{\bar{X}-e^*}{T}\right)$, the lower bond of $\hat{C}$ and C
- y* = $\frac{\bar{X}-e^*}{T}$, $x^* = h^{-1}(y^*) = X_0^A$ (12_20 pg.12)
- discuss 2 cases :
  - case 1 : y* satisfy $\hat{g}$(y$^*$) = g($y^*$) , can be regarded as Type A
  - case 2 : y* satisfy $\hat{g}(y^*)$ < g($y^*$)

# Case 1 of Type B ($\hat{g}$(y$*$ ) = g(y$*$))

recall Type A strategy :

- define the range of $E_T^A$:
  - $k(x) \triangleq x + h(x)T$ (total trade volume before last lump purchase $\Delta X_T^A$)
  - there exists a unique $\bar{e} \in (0, \bar{X})$ such that k($\bar{e}$) = $\bar{X}$
  - therefore, feasible strategy of Type A : $0 \leq X_0^A \leq \bar{e}$
  - $\because E_T^A = \bar{X} - h\left(X_0^A\right)T, \therefore \bar{e} \leq E_T^A \leq \bar{X}$
- the minimization problem we consider :
  - G(e) $= \phi(\mathrm{e}) + Tg(\frac{\bar{X}-e}{T})$
- therefore, the minimum of convex function G over[0, $\bar{X}$] is obtained in [$\bar{e}$, $\bar{X}$]
- but here $\hat{\mathrm{g}}(\bar{\mathrm{e}}) \leq$ g($\bar{\mathrm{e}}$), so we need to prove **e*** $\geq$ $\boldsymbol{\bar{e}}$

$y^* = \frac{\bar{X}-e^*}{T}$, $x^* = h^{-1}(y^*)$

$k(x) \triangleq x + h(x)T$

$k(\bar{e}) = \bar{X}$

$\hat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X}-e}{T}\right)$

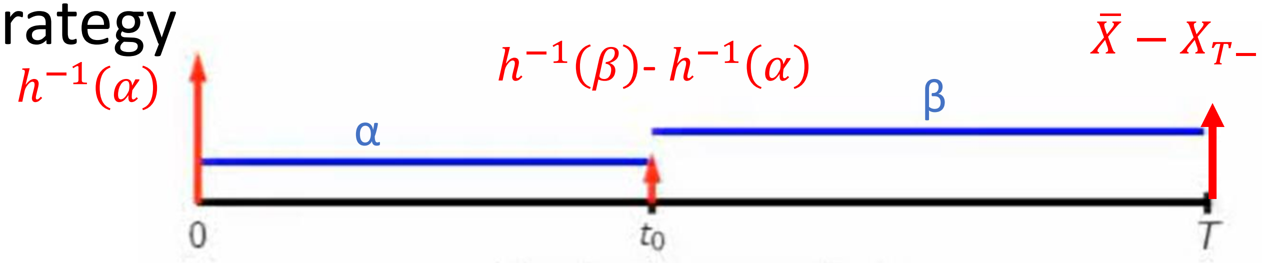
# Case 1 of Type B

- case $x^* = 0$ :
  - $y^* = 0$, $e^* = \bar{X}$
- case $x^* > 0$ :
  - subcase $0 < x^* \leq F(0+)$ (1/17_pg. 19-20)
  - subcase $x^* > F(0+)$ (1/17_pg. 21-22)
- conclusion:
  - when $\hat{g}(y*) = g(y*)$, we can use the Type A strategy
  - with $X_0^A = x^*$ and $E_T^A = e^*$

$$E_t = X_t - \int_0^t h(E_s)ds,$$

# Case 2 of Type B

- $\hat{g}(y^*) < g(y^*)$
- define $t_0 \in (0, T)$ by $t_0 = \frac{(\beta - y^*)T}{\beta - \alpha}$
  - so that $\alpha t_0 + \beta(T - t_0) = y^* T$
- consider the Type B strategy that :
  - makes an initial strategy $X_0^B = h^{-1}(\alpha)$
  - then purchase at rate $\mathrm{d}X_t^B = \alpha dt$, for $0 \le t < t_0$ $\left(\mathbf{E_t^B} = \mathbf{h^{-1}}(\boldsymbol{\alpha})\right)$
  - follow with the purchase $\Delta X_{t0}^B = h^{-1}(\beta) - h^{-1}(\alpha)$ at time $t_0$
  - then purchase at rate $\mathrm{d}X_t^B = \beta dt$, for $t_0 \le t < T$ $\left(\mathbf{E_t^B} = \mathbf{h^{-1}}(\boldsymbol{\beta})\right)$
  - final lump purchase $\bar{X} - X_{T-}^{\beta}$ at time T
- we will show that $\mathrm{X^B}$ is the optimal strategy



$$t_0 = \frac{(\beta - y^*)T}{\beta - \alpha} \rightarrow \alpha t_0 + \beta(T - t_0) = y^* T$$

y* = $\frac{\bar{X} - e^*}{T}$

# Case 2 of Type B

- 2 parts of the proof
  - $\boldsymbol{X^{\mathbf{B}}}$ **is optimal** with : $X_t^B$ = $\begin{cases} h^{-1}(\alpha) + \alpha t, & 0 \leq t < t_0, \\ h^{-1}(\beta) + \alpha t_0 + \beta(t - t_0), & t_0 \leq t < T \\ \bar{X}, & t = \mathrm{T} \end{cases}$
  - $\Delta X_T^B = \bar{X} - h^{-1}(\beta) - \alpha t_0 - \beta(t - t_0)$
    $= \bar{X} - \mathrm{h}^{-1}(\beta) - \mathrm{y}^*\mathrm{T}$
    $= e^* - h^{-1}(\beta)$

  also need to prove $\boldsymbol{h^{-1}(\beta)} \leq \boldsymbol{e}^*$

C(X) $= \phi(E_T) + \int_0^T g(h(E_t))dt$

# $X^B$ is optimal

- $\hat{C}(X) \triangleq \phi(E_T) + \int_0^T \hat{g}\big(h(E_t)\big)dt$
- we obviously have $\hat{C}(X) \leq C(X)$
- By Jensen's Inequality (12_20 pg, 14) :
  - $\hat{C}(X) \geq \phi(E_T) + T\hat{g}\left(\frac{\bar{X}-E_T}{T}\right)$
- This lead us to consider minimization of the function $\widehat{G}$ :
  - $\widehat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X}-e}{T}\right)$
- prove that :
  - $C(X^B) = \widehat{G}(e^*)$

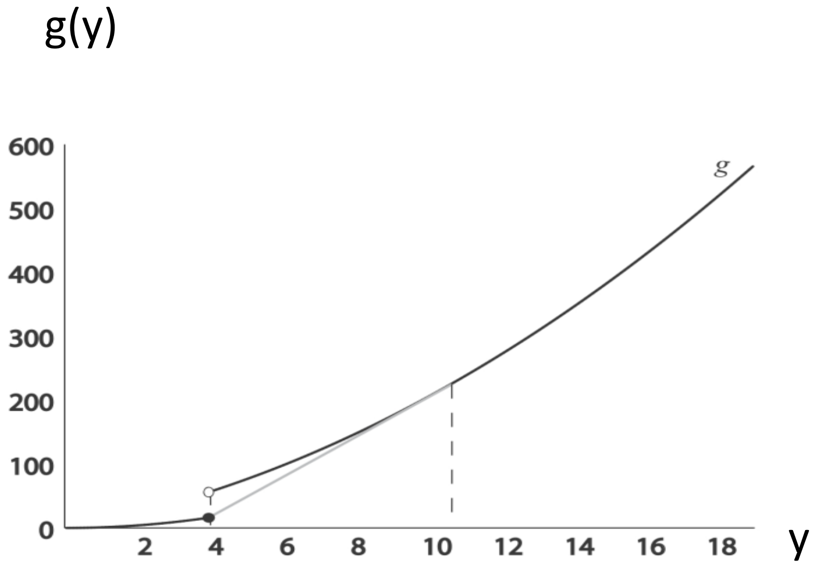
# $X^B$ is optimal

$$\Delta X_T^B = \bar{X} - h^{-1}(\beta) - \alpha t_0 - \beta(t - t_0)$$
$$= \bar{X} - h^{-1}(\beta) - y^* T$$
$$= e^* - h^{-1}(\beta)$$

$$t_0 = \frac{(\beta - y^*)T}{\beta - \alpha} \rightarrow \alpha t_0 + \beta(T - t_0) = y^* T$$

- $E_T^B = E_{T-}^B + \Delta E_T^B = h^{-1}(\beta) + \Delta X_T^B = e^*$
- $C(X^B) = \phi(E_T^B) + \int_0^T g\left(h\left(E_t^B\right)\right) dt$

$$= \phi(e^*) + g(\alpha)t_0 + g(\beta)(T - t_0)$$
$$= \phi(e^*) + l(\alpha)t_0 + l(\beta)(T - t_0)$$
$$= \phi(e^*) + Tl\left(\frac{\alpha t_0 + \beta(T - t_0)}{T}\right)$$
$$= \phi(e^*) + Tl(y^*)$$
$$= \phi(e^*) + T\hat{g}(y^*)$$
$$= \hat{G}(e^*)$$




if $y^* \in (0, \bar{Y})$ which satisfies $\hat{g}(y^*) < g(y^*)$, then exists unique l below g : $0 \le \alpha < y^* < \beta \le \bar{Y}$

$l(\alpha) = \hat{g}(\alpha) = g(\alpha)$, $l(\beta) = \hat{g}(\beta) = g(\beta)$

$l(y) = \hat{g}(y) < g(y)$, $\alpha < y < \beta$

$g(y) \triangleq y\Psi(h^{-1}(y))$

$y^* = \frac{\bar{X}-e^*}{T}$

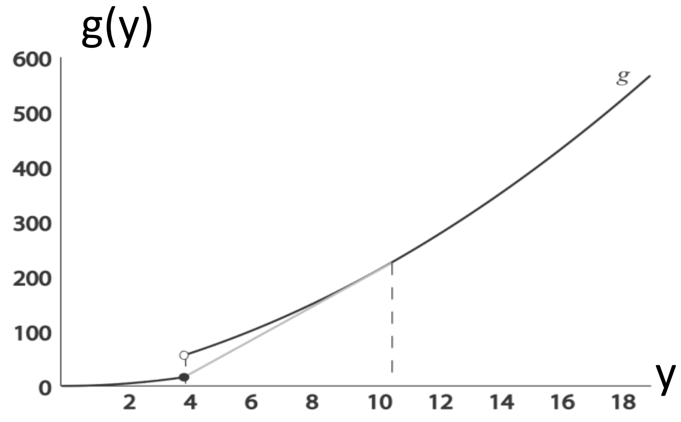
$\hat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X}-e}{T}\right)$

Cost func 的 lower bond

# $h^{-1}(\beta) \leq e^*$

- for all $e \in (e^*, \bar{X})$, $D^+\hat{G}(e) > 0$
- assume e is greater than but sufficiently close to e*:
  - $\frac{\bar{X}-e}{T}$ is in $(\alpha, y^*)$, $\because \hat{g}(y^*) < g(y^*)$ and $y^* = \frac{\bar{X}-e^*}{T}$
  - where $\hat{g}$ is linear with slope $\frac{g(\beta)-g(\alpha)}{\beta-\alpha}$, $\because$ l(y) = $\hat{g}(y) < g(y)$, $\alpha < y < \beta$
- proof :

$$
\begin{aligned}
0 &< D^+\hat{G}(e) \\
&= D^+\Phi(e+) - D^-\hat{g}(y)\Big|_{y=\frac{X-e}{T}} \\
&= \psi(e+) - \frac{g(\beta)-g(\alpha)}{\beta-\alpha} \\
&= \psi(e+) - \frac{\beta\psi(h^{-1}(\beta)) - \alpha\psi(h^{-1}(\alpha))}{\beta-\alpha} \\
&\leq \psi(e+) - \frac{\beta\psi(h^{-1}(\beta)) - \alpha\psi(h^{-1}(\beta))}{\beta-\alpha} \\
&= \psi(e+) - \psi(h^{-1}(\beta)).
\end{aligned}
$$




- $\psi(e+) > \psi(h^{-1}(\beta))$ for all greater than but sufficiently close to e* implies $h^{-1}(\beta) \leq e^*$

# Example

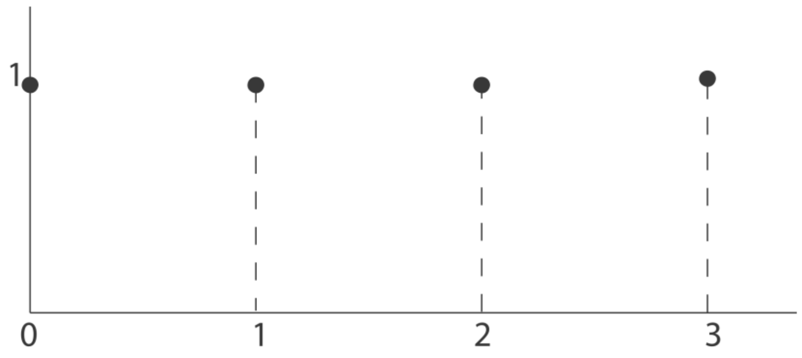
Discrete order book

# Example 3 (Discrete order book)

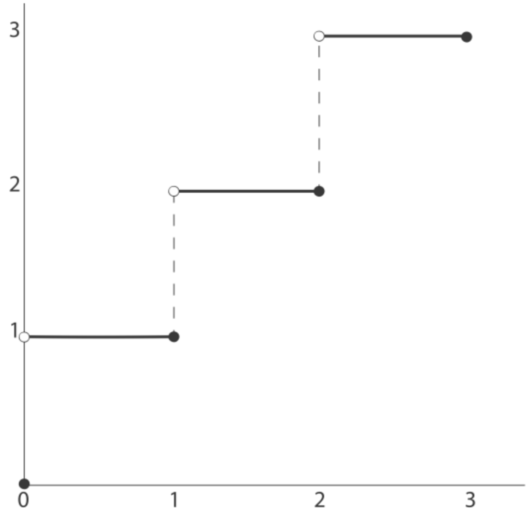
- F(x) = $\sum_{i=0}^{\infty} I_{(i,\infty)}\,(\mathrm{x}), \quad \mathrm{x} \geq 0$
- Ψ(y)= $\sum_{i=1}^{\infty} I_{(i,\infty)}\,(\mathrm{y}), \quad \mathrm{y} \geq 0$
- where F(j) = j, F(j+) = j+1, Ψ(j+1) = j, Ψ(j+) = j
- F(Ψ(j)+ ) = j , Ψ(F(j)+) = j

- **for $\mathbf{k \geq 1}$ and $\mathbf{k < y \leq k + 1}$, Ψ(y)= k**

density




F(x)

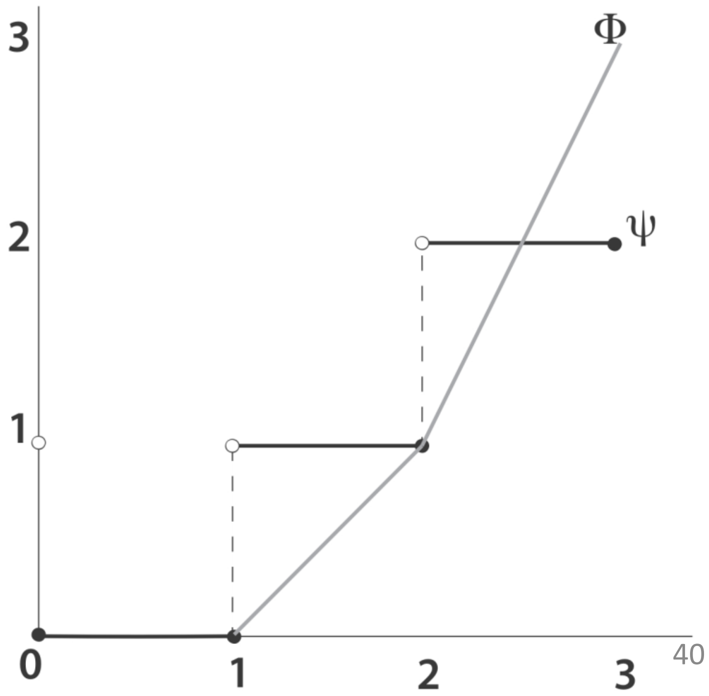


$\rho(x) \triangleq \int_{[0,x)} \xi dF(\xi), \qquad x \geq 0$

$\phi(y) \triangleq \rho(\Psi(y)) + [y - F(\Psi(y))]\Psi(y)$

# Example 3 (Discrete order book)

- $\rho(x) = \sum_{i=0}^{\infty} iI_{(i,\infty)}(x)$
  - in particular, $\rho(0) = 0$
  - and for integers $k \geq 1$ and $k - 1 < x \leq k$, $\rho(x) = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}$
- $\phi(y) = \rho(\Psi(y)) + [y - F(\Psi(y))]\Psi(y)$
  - $\rho(\Psi(y)) = \frac{k(k-1)}{2}$ ($\Psi(y) = k$, for $k < y \leq k+1$)
  - lump purchase : $[y - F(\Psi(y))]\Psi(y) = k(y-k)$
  - we get : $\phi(y) = \frac{k(k-1)}{2} + k(y-k)$
- $\phi$ is convex, with differential :
  - $\partial\phi(y) = [\Psi(y), \Psi(y+)]$, for all $y \geq 0$
  - $\phi'(y) = \Psi(y) = k$, for all $y \geq 0$



# Example of Type B (Discrete order book)

- In order to illustrate different cases of purchasing strategies, we assume :
  - $h(x) = x$ and T = 1
- The function $\hat{G}$ is minimized over to $[0, \bar{X}]$ at e* if and only if :
  - $\mathbf{0} \in \boldsymbol{\partial \hat{G}}(\boldsymbol{e}^*) = \boldsymbol{\partial \phi}(\boldsymbol{e}^*) - \boldsymbol{\partial \hat{g}}(\boldsymbol{\bar{X}} - \boldsymbol{e}^*)$
  - which is equivalent to $\partial \phi(e^*) \cap \partial \hat{g}(\bar{X} - e^*) \neq \emptyset$

$$\hat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X} - e}{T}\right)$$

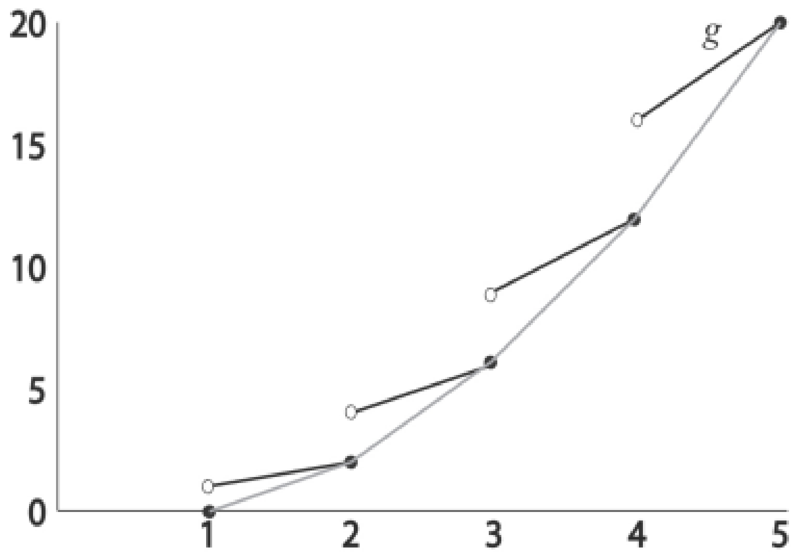
$g(y) \triangleq y\Psi(h^{-1}(y))$

$\Psi$(y)= $\sum_{i=1}^{\infty} I_{(i,\infty)}$ (y)

**for $\mathbf{k \geq 1}$ and $\mathbf{k < y \leq k+1}$, $\Psi$(y)= k**

# convex hull and affine function

g(y)




y

1.

if $y^* \in (0, \overline{\mathrm{Y}})$ which satisfies $\hat{g}(y^*) < g(y^*)$, then exists unique l below g :

$0 \leq \alpha < y^* < \beta \leq \overline{Y}$

l($\alpha$) = $\hat{g}(\alpha)$ = g($\alpha$), l($\beta$) = $\hat{g}(\beta)$ = g($\beta$)

l(y) = $\hat{g}(y)$ < g(y), $\alpha < y < \beta$

**For $\boldsymbol{\alpha} < \boldsymbol{y}^* < \boldsymbol{\beta}$, we have $\mathbf{l}(\mathbf{y}^*) = \hat{\boldsymbol{g}}(\boldsymbol{y}^*) < \boldsymbol{g}(\boldsymbol{y}^*)$**

2.

$g(y) = ky$, for integer $k \geq 0$ $and$ $k < y \leq k+1$

in particular, $g(k) = (k-1)k$

3.

The convex hull of g interpolates linearly between (k,(k-1)k) and (k+1,k(k+1)), i.e. $\hat{g}(y) = k(2y - (k+1))$

$\phi(y) = \frac{k(k-1)}{2} + k(y-k) = k\left(y - \frac{1}{2}k - \frac{1}{2}\right)$

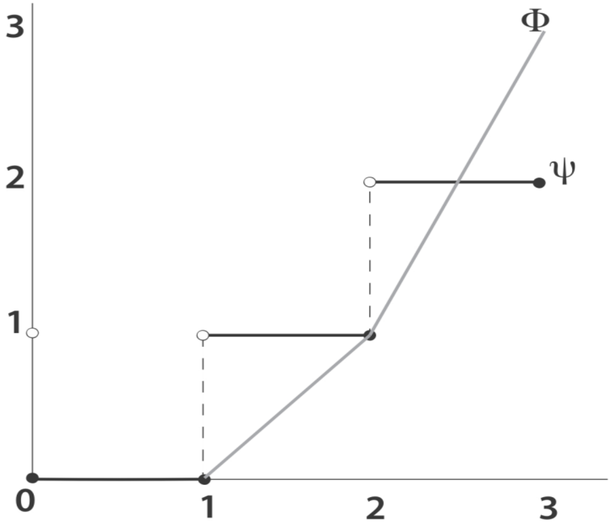
$\hat{g}(y) = k(2y - (k+1))$

# Example of Type B (Discrete order book)

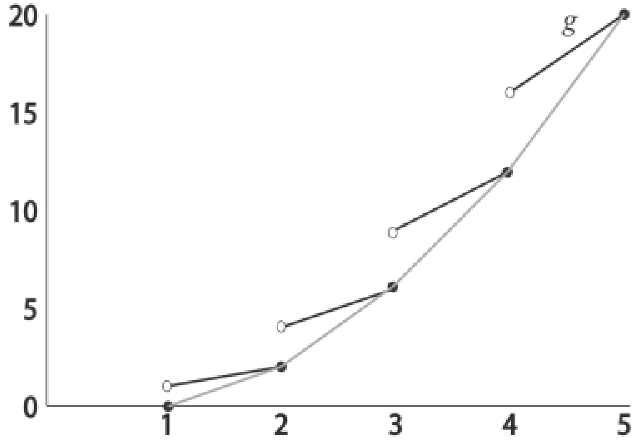
- find e* where $\partial\phi(e^*) \cap \partial\hat{g}(\bar{X} - e^*) \neq \emptyset, and$ for integers $k \geq 0$ :

- $\partial\phi(y) = \begin{cases} \{0\}, & y = 0 \\ [k-1, k], & y = k \\ \{k\}, & k < y < k+1 \end{cases}$

- $\partial\hat{g}(y) = \begin{cases} \{0\}, & y = 0 \\ [2(k-1), 2k], & y = k \\ \{2k\}, & k < y < k+1 \end{cases}$




g(y)




y

$$\hat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X} - e}{T}\right)$$

# Example of Type B (Discrete order book)

- we assume :
  - $h(x) = x$
  - T = 1
- we define k*to be the largest integer less than or equal to $\frac{\bar{X}}{3}$, so that :
  - $3k^* \leq \bar{X} < 3k^* + 3$
- we divide the analysis into three cases:
  - Case A : $3k^* \leq \bar{X} < 3k^* + 1$
  - Case B : $3k^* + 1 \leq \bar{X} < 3k^* + 2$

$\hat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X} - e}{T}\right)$

# Case A

- Case A : $3k^* \leq \bar{X} < 3k^* + 1$
- we define e* = $\bar{X} - k^*$, so that $2k^* \leq e^* < 2k^* + 1$ and k*= $\bar{X} - e^*$
- then :
  - $\partial\phi(e*) \ni 2k^*$
  - $\partial\hat{g}(\bar{X} - e^*) = [2(k^* - 1), 2k^*]$
- so the intersection of $\partial\phi(e*)$ and $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired, with e* = $\bar{X} - k^*$

$$\partial\phi(y) = \begin{cases} \{0\}, & y = 0 \\ [k-1, k], & y = k \\ \{k\}, & k < y < k+1 \end{cases}$$

$$\partial\hat{g}(y) = \begin{cases} \{0\}, & y = 0 \\ [2(k-1), 2k], & y = k \\ \{2k\}, & k < y < k+1 \end{cases}$$

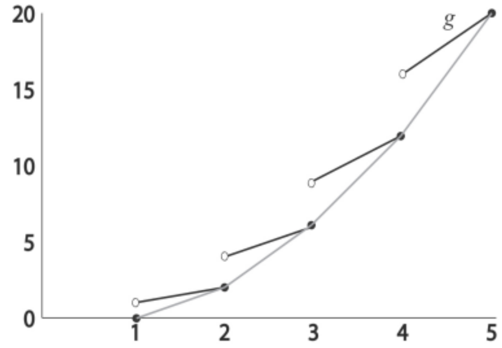
# Case A

$y^* = \frac{\bar{X}-e^*}{T}$

$x^* = h^{-1}(y^*) = X_0^A$

$$E_t = X_t - \int_0^t h(E_s)ds,$$

- we have e* = $\bar{X} - k^*$
  - $x^* = y^* = k^*$ (an integer)
  - $\hat{g}(y^*) = g(y^*)$, which is Type A strategy
- two cases:
  - if k* = 0, first subcase of Case1 (x<F(0+)):
    - x* = k* = 0, initial lump purchase : 0 shares
    - do nothing until time T
    - with final lump purchase T : $\bar{X}$
  - if k* > 0:
    - x* = k* > 0, initial lump purchase : k* shares
    - purchase continuously at rate k* in (0,T), to keep $E_t = k^*$ *and* $D_t = \Psi(E_t)$ = k*-1
    - with final lump purchase T : $\bar{X} - 2k^*$



$\hat{G}(e) = \phi(e) + T\hat{g}\left(\frac{\bar{X} - e}{T}\right)$

# Case B

- Case B : $3k^* + 1 \leq \bar{X} < 3k^* + 2$
- we define e* = $2k^* + 1$, so that $k^* < \bar{X} - e^* < k^* + 1$
- then :
  - $\partial\phi(e*) = [2k^*, 2k^* + 1]$
  - $\partial\hat{g}(\bar{X} - e^*) \ni 2k^*$
- so the intersection of $\partial\phi(e*)$ and $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired, with e* = $2k^* + 1$

$$\partial\phi(y) = \begin{cases} \{0\}, & y = 0 \\ [k-1, k], & y = k \\ \{k\}, & k < y < k+1 \end{cases}$$

$$\partial\hat{g}(y) = \begin{cases} \{0\}, & y = 0 \\ [2(k-1), 2k], & y = k \\ \{2k\}, & k < y < k+1 \end{cases}$$

# Case B

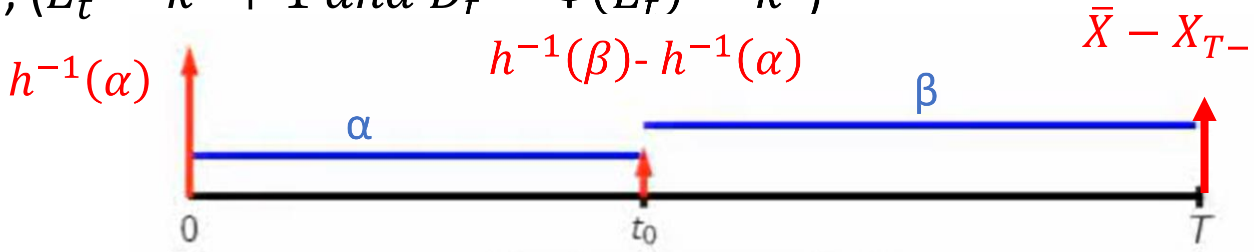
$y^* = \frac{\bar{X}-e^*}{T}$

$x^* = h^{-1}(y^*) = X_0^A$

$E_t = X_t - \int_0^t h(E_s)ds,$

$\Psi(y) = \sum_{i=1}^{\infty} I_{(i,\infty)}(y)$

- we have e* = $2k^* + 1$
  - $x^* = y^* = \bar{X} - e^*$
  - $k^* \leq y^* < k^* + 1,\ so\ \hat{g}(y^*) < g(y^*)$, which is Type B strategy
- By Type B strategy, we have :
  - $\alpha = k^*$
  - $\beta = k^* + 1$
  - $t_0 = \frac{(\beta - y^*)T}{\beta - \alpha} = (\beta - y^*) = k^* + 1 - \bar{X} - e^* = 3k^* + 2 - \bar{X}$
- the strategy :
  - initial lump purchase : $X_0^B = h^{-1}(\alpha)$ = k*
  - purchases continuously at rate k* in (0, $t_0$), ($E_t = k^*\ and\ D_t = \Psi(E_t) = k^* - 1$)
  - intermediate lump purchase : $\Delta X_{t0}^B = h^{-1}(\beta) - h^{-1}(\alpha) = \beta - \alpha = 1$
  - purchases continuously at rate k*+1 in ($t_0$,1), ($E_t = k^* + 1\ and\ D_t = \Psi(E_t) = k^*$)
  - final lump purchase : $X_T^B = k^*$



# implementation

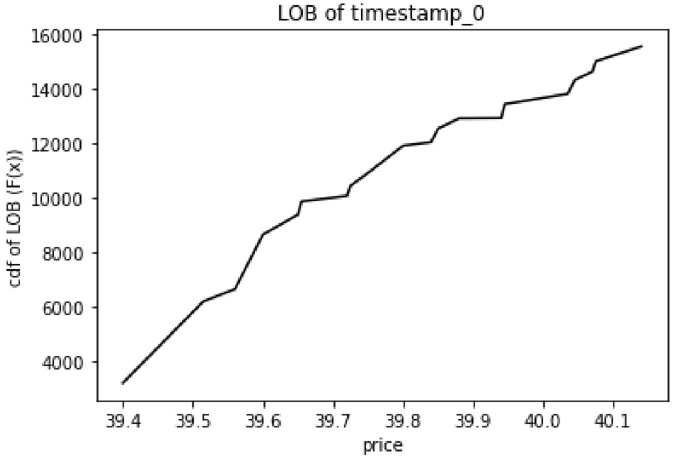
Adidas, 7/1 (simple)

# Measure of LOB

- Adidas, 7/1, ask-side LOB (partial)

| Date | Ask1 | Ask2 | Ask3 | Ask4 | Ask5 | Ask6 | Ask7 | Ask8 | Ask9 | Ask10 | Ask11 | Ask12 | Ask13 | Ask14 | Ask15 | Ask16 | Ask17 | Ask18 | Ask19 | Ask20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 32579015 | 3.94E+01 | 3.95E+01 | 3.96E+01 | 3.96E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.98E+01 | 3.98E+01 | 3.98E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 4.00E+01 | 4.00E+01 | 4.01E+01 | 4.01E+01 | 4.01E+01 |
| 32579017 | 3.94E+01 | 3.95E+01 | 3.96E+01 | 3.96E+01 | 3.96E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.98E+01 | 3.98E+01 | 3.98E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 4.00E+01 | 4.00E+01 | 4.01E+01 | 4.01E+01 |
| 32579021 | 3.94E+01 | 3.94E+01 | 3.95E+01 | 3.96E+01 | 3.96E+01 | 3.96E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.98E+01 | 3.98E+01 | 3.98E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 4.00E+01 | 4.00E+01 | 4.01E+01 |
| 32579070 | 3.94E+01 | 3.94E+01 | 3.96E+01 | 3.96E+01 | 3.96E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.98E+01 | 3.98E+01 | 3.98E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 4.00E+01 | 4.00E+01 | 4.01E+01 | 4.01E+01 |
| 32579071 | 3.94E+01 | 3.94E+01 | 3.96E+01 | 3.96E+01 | 3.96E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.98E+01 | 3.98E+01 | 3.98E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 4.00E+01 | 4.00E+01 | 4.01E+01 | 4.01E+01 |
| 32579072 | 3.94E+01 | 3.94E+01 | 3.96E+01 | 3.96E+01 | 3.96E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.97E+01 | 3.98E+01 | 3.98E+01 | 3.98E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 3.99E+01 | 4.00E+01 | 4.00E+01 | 4.01E+01 |

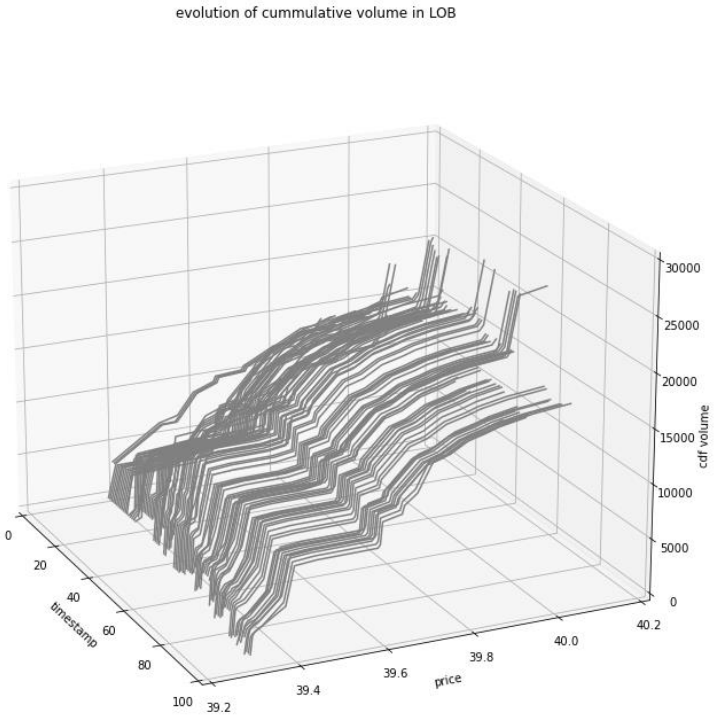
| Date | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 | Q9 | Q10 | Q11 | Q12 | Q13 | Q14 | Q15 | Q16 | Q17 | Q18 | Q19 | Q20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 32579015 | 3174 | 3000 | 452 | 2000 | 731 | 483 | 205 | 373 | 469 | 1000 | 132 | 496 | 374 | 12 | 509 | 379 | 511 | 305 | 379 | 545 |
| 32579017 | 3174 | 3000 | 50 | 452 | 2000 | 731 | 483 | 205 | 373 | 469 | 1000 | 132 | 496 | 374 | 12 | 509 | 379 | 511 | 305 | 379 |
| 32579021 | 93 | 3174 | 3000 | 50 | 452 | 2000 | 731 | 483 | 205 | 373 | 469 | 1000 | 132 | 496 | 374 | 12 | 509 | 379 | 511 | 305 |
| 32579070 | 93 | 3174 | 50 | 452 | 2000 | 731 | 483 | 205 | 373 | 469 | 1000 | 132 | 496 | 374 | 12 | 509 | 379 | 511 | 305 | 379 |
| 32579071 | 93 | 3174 | 50 | 452 | 2000 | 731 | 483 | 205 | 373 | 469 | 1000 | 132 | 496 | 374 | 12 | 509 | 379 | 511 | 4908 | 379 |
| 32579072 | 93 | 3174 | 50 | 452 | 2000 | 731 | 483 | 2500 | 205 | 373 | 469 | 1000 | 132 | 496 | 374 | 12 | 509 | 379 | 511 | 4908 |



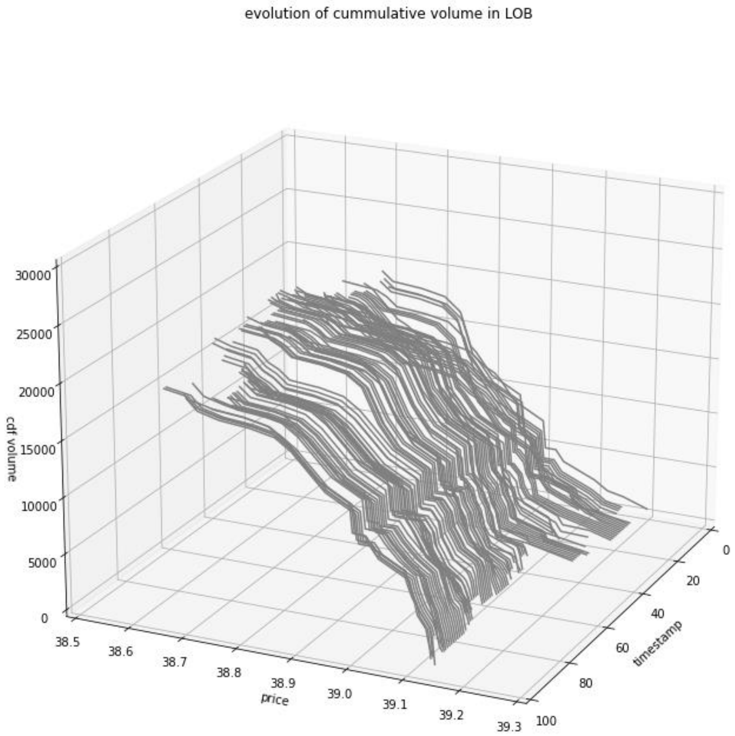
# Measure of LOB

- Evolution of cumulative volume in LOB

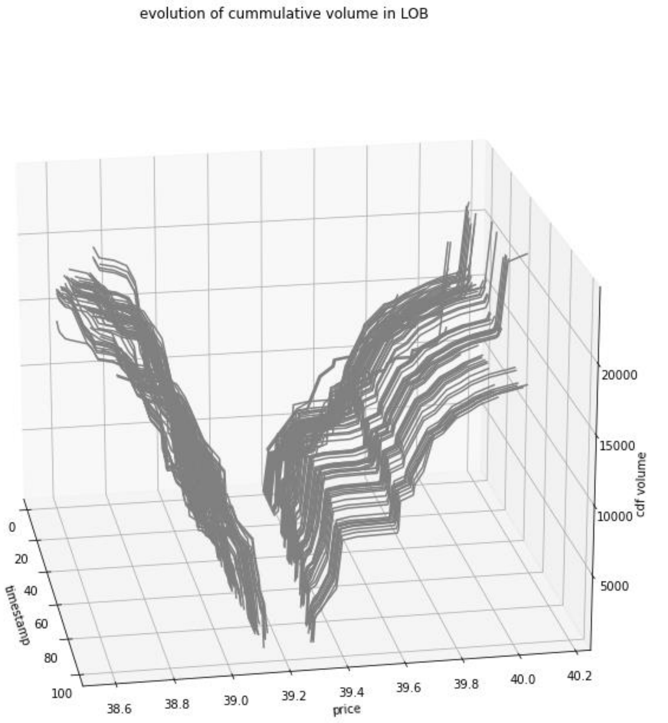
Ask side




Bid side




both side



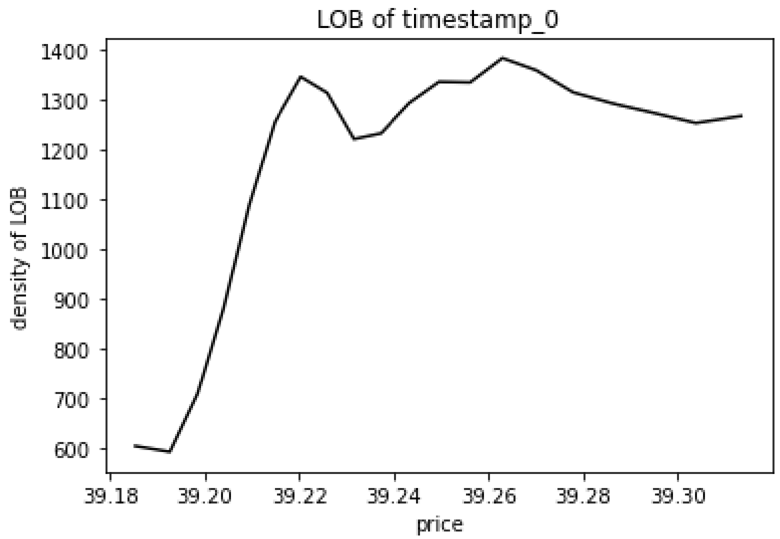

* only first 100 timestamps of 7/1

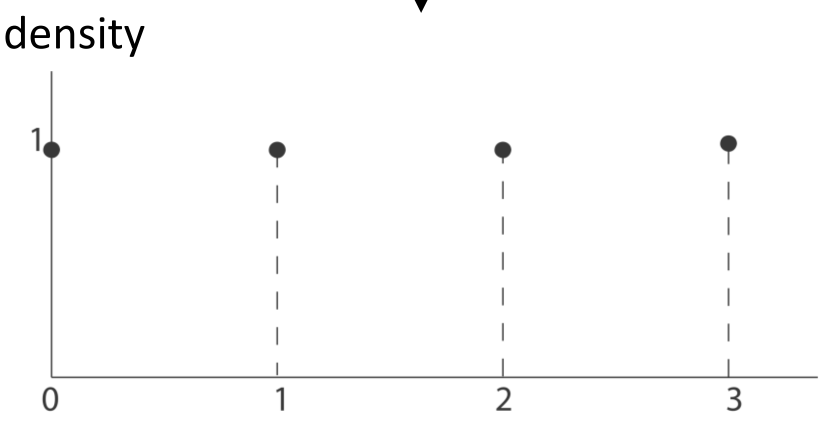
# Measure of LOB

- observation
  - less market impact, less purchasing cost
- implementation
  - Adidas, 7/1, ask-side LOB
  - use ‘’Discrete order book’’ (Example 3 in paper) as a simple example
  - fixed average volume with discrete price

# implementation

- assume $\mathrm{T} = 1, \mathrm{h}(\mathrm{x}) = \mathrm{x}$
- $A_{0-}$= 3918 (39.18*100)
  - price (x) → 100*price
  - 3918, 3919, 3920…
- density ~ 1167 (average volume of ask-side LOB)
  - volume (y, $\bar{X}$) → volume/1167
- we assume :
  - $\bar{X} = 4 * 10^3$ (1167 shares)

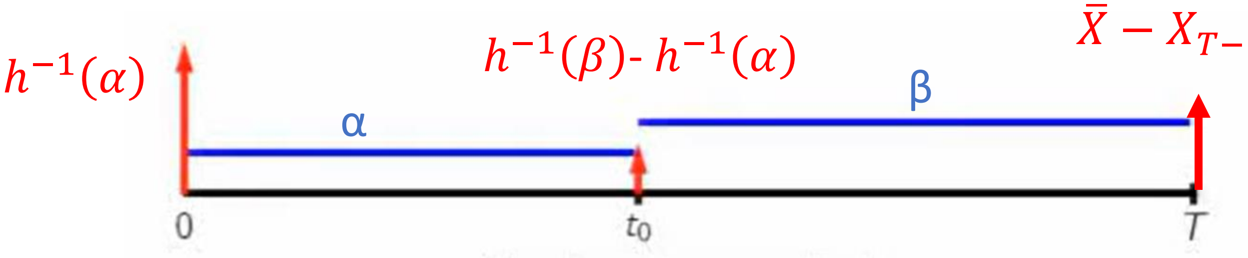



* average of 84985 LOB data

# implementation

- $\bar{X} = 4 * 10^3$ (1167 shares)
- we define k*to be the largest integer less than or equal to $\frac{\bar{X}}{3}$, so that :
  - $3k^* \le \bar{X} < 3k^* + 3$
  - k* = 1333 and $\bar{X}$ = $3k^* + 1$
- Case B : $3k^* + 1 \le \bar{X} < 3k^* + 2$
- $\alpha = k^* = 1333$
  - $\beta = k^* + 1 = 1334$
  - $t_0 = 3k^* + 2 - \bar{X} = 1$
- strategy
  - $X_0^A = 1333$
  - purchase continuously at rate k* = 1333
  - $X_T^A = \bar{X} - 2k^* = 1334$



# performance

- strategy
  - $X_0^A = 1333$
  - purchase continuously at rate k* = 1333
  - $X_T^A = \bar{X} - 2k^* = 1334$ with $E_T^A$ = 2667
- comparison
  - $X_0^A = 1000$
  - purchase continuously at rate k* = 1000
  - $X_T^A = \bar{X} - 2k^* = 1000$ with $E_T^A = 3000$
- performance (ignore fixed costs)
  - our strategy : C(X) = ϕ$(2667)$ +$Tg(1333) =$ **5330667**
  - comparison : C(X) = ϕ$(3000)$ +$Tg(1000) =$ **5497500**

# Conclusion

- Type A strategy is one kind of Type B strategy with the intermediate purchase is of size 0
- with density or cdf of LOB and resilience pattern :
  - we can derive the cost function
  - if the cost function is convex : Type A strategy
  - if the cost function isn’t convex : Type B strategy
- no risk aversion in this model
  - only focus on minimizing E(C(X)) of the execution
- no discussion about how to use the model in practice
  - measure of F (cdf of LOB), h(resilience function)
- no discrete-time version
  - unlike Obizhaeva & Wang (2005) , Alfonsi, Fruth and Schied (2010)

# Ref.


- https://www.pathlms.com/siam/courses/2725/sections/3581/video_presentations/29246
- https://www.math.cmu.edu/users/shreve/OptimalExecution.pdf